Symmetries of the Relativistic Hydrogen Atom*†

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The structure of the symmetry group of the relativistic hydrogen atom (no spins) is determined, in order to study how a symmetry group which places particles of different spin in the same supermultiplet can be incorporated into a relativistic theory. The difficulties associated with the extension of the symmetry group of the bound states to bound-state scattering are discussed.

I. INTRODUCTION

Since the first days of quantum mechanics, it has been known that the energy levels of the hydrogen atom possess an extra degeneracy beyond that given by rotational invariance; states of different angular momentum but with the same principal quantum number have the same energy. This phenomenon has long been understood in terms of symmetries: The Coulomb problem admits, in addition to the angular momentum, a second conserved vector, the Lenz vector. (In the classical problem, the Lenz vector points to the perihelion of the orbit; its constancy in time is simply the statement that Keplerian orbits do not precess.) The angular momentum, together with the Lenz vector (divided by the square root of the absolute value of the Hamiltonian), form a set of generators for the group $O(4)$; the states of a given energy form a basis for a single irreducible representation of this group.

In 1954, Cutkosky[1] analyzed a relativistic generalization of this problem; he solved the ladder approximation to the Bethe-Salpeter equation[2] for two massive spinless particles exchanging a massless spinless meson. He found that the bound states of this system possess the same $O(4)$ symmetry as in the nonrelativistic problem.

This result stirred no furor. However, in the aftermath of the attempts to construct a relativistic version of the $SU(6)$ theory,[3] there were left a number of theorems[4] which denied the possibility that in a physically sensible theory relativistic invariance and an internal symmetry could be combined in any but the most trivial way. A basic characteristic of theories like relativistic $SU(6)$, and one which is specifically excluded by some of the "impossibility theorems,"[5] is that the symmetry places degenerate particles of different spin in the same multiplet.

How can we reconcile these theorems with the Cutkosky problem? Which hypotheses does the Cutkosky problem not satisfy? What has the Cutkosky problem to tell us about the role of symmetries like $SU(6)$ in relativistic physics?

In Sec. II we initiate this investigation by recapitulating some of Cutkosky's results. We rotate the Bethe-Salpeter equation into Euclidean space, carry out Cutkosky's stereographic projection of Euclidean momentum space onto the surface of a four-sphere, and explicitly display the generators of the $O(4)$ invariance group.

In Sec. III we attempt to find a group of symmetries which contains the Cutkosky and (Euclidean) space-time invariances. To be precise, we attempt to find a group of transformations with the following characteristics:

(1) It turns solutions of the Euclidean-space Bethe-Salpeter equation into other solutions of the equation with the same mass.

(2) It contains the four-dimensional Euclidean group.

(3) It is a finite-parameter Lie group.

(4) It has the property that, in the subspace of states of any given four-momentum, there are transformations in the group which, when restricted to that subspace, reproduce the effect of the $O(4)$ symmetry group of Cutkosky.

(5) It is simply expressible in terms of the total and relative (Euclidean) four-momenta. By this we mean that its generators are expressible as differential operators in these variables. To avoid infinite-order differential generators, and to maintain condition (3), we restrict ourselves from the beginning to first-order differential operators.

It is clear that condition (3) is the heart of the problem; without it we could solve the problem trivially. A solution would be the group whose generators are the generators of the Euclidean group plus generators which, acting on states of one fixed four-

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momentum, generate transformations identical to the $O(4)$ Cutkosky transformations, and which when acting on states of a different four-momentum, generate the identity transformation.

We classify transformations satisfying (1) and (5) according to the transformation properties of their generators under the homogeneous Euclidean group. We find two groups which satisfy all our conditions. In one case the generators transform according to the $(1,0)\oplus(1,0)$ representation of the homogeneous Euclidean group, in the other case according to $(0,1)\oplus(0,1)$. Both groups commute with the generators of space translations, and they are turned into each other by space reflection. The union of the two groups produces an infinite-parameter group.

In Sec. IV we construct the generators of the Cutkosky transformations as first-order differential operators and calculate their commutation relations. We find that the two groups are isomorphic and that they each have the structure of the semidirect product of the four-dimensional Euclidean group with $O(4)$. We then show that the little group for fixed four-momentum has the structure $O(3)\otimes O(4)$, with the $O(3)$ factor being represented trivially on states of the given four-momentum, precisely as in the case of the nonrelativistic hydrogen atom.

In Sec. V, we perform the inverse Wick rotation in the equation and group generators back to Minkowski space. We explicitly demonstrate that the analytic continuations of our group generators generate invariances of the Minkowski-space Bethe-Salpeter equation. The complications induced by the noncompactness of the Lorentz group show up in this analytic continuation. The $(1,0)\oplus(1,0)$ representation of $O(4)$ is real, but the $(1,0)\oplus(1,0)$ representation of the homogeneous Lorentz group is complex, and so to make the symmetry group Lorentz-complete we must add six new generators; the new ones are just i times the original generators. The symmetry group in Minkowski space is a semidirect product of the Poincaré group with $O(4,C)$. The commutators of the generators of accelerations with the generators of $O(4,C)$ ensure that the $O(4,C)$ group is composed of nonunitary operators.

When we examine the little group for a fixed four-momentum, we find that it contains operators which appear unitary when restricted to the invariant subspace of bound states of the given four-momentum. (The field-theory Hilbert-space normalization for the bound states gives rise to a well-defined norm for the Minkowski-space Bethe-Salpeter wave functions. In the Appendix the form of such a norm, valid for degenerate solutions of the equation, is derived, and the "apparently unitary" transformations are shown to be unitary with respect to it.) This unitary subgroup of the little group has exactly the same structure as the Euclidean-space little group, and so we recover the $O(4)$ nonrelativistic hydrogen-atom symmetry group from the Minkowski-space Bethe-Salpeter equation.

In Sec. VI we extend Cutkosky's analysis to the unbound states of the Bethe-Salpeter equation. The analytic continuation from below to above threshold leaves the structure of the full symmetry group unchanged, but the unitary little group is modified to $O(3)\otimes O(3,1)$, with the $O(3)$ represented trivially. This is just the same as the symmetry group of the scattering states in the nonrelativistic Coulomb problem.

At this point we have learned how Cutkosky's problem escapes the strictures of some of the "impossibility theorems." The theorems which showed that infinite numbers of degenerate particles occur in theories like relativistic $SU(6)$ relied on the facts that in such theories the little group is noncompact and is represented unitarily. Yet we can avoid this in a physically sensible theory: The little group in Cutkosky's problem is still noncompact, but only a *compact subgroup* is represented unitarily. Hence there are no infinite-particle-number troubles.

It might seem that we need only abstract the group structure from the Cutkosky problem and then go ahead and build a relativistic $SU(6)$ theory, but this is a false idyll. In Sec. VII we show that if we were to extend the Cutkosky symmetry group to the scattering of bound states, then there would be no scattering. The argument is not restricted to the Cutkosky problem, and depends only on those very unitarity properties of the group elements which were crucial in obtaining a spectrum with a finite number of particles.

Although in the Cutkosky problem the particle spectrum displays $O(4)$ symmetry, the scattering of bound states does not have this symmetry. Nonetheless, the Cutkosky problem illustrates one way that a symmetry like $SU(6)$ can be relevant to relativistic physics, yet consistent with the fact that the only possible symmetry groups of a nontrivial S matrix are direct products of the Poincaré group with an internal symmetry.

II. PRELIMINARIES—THE CUTKOSKY EQUATION

We will begin our investigation of the symmetries of Cutkosky's Bethe-Salpeter equation by reviewing his solution. The equation is the ladder approximation to the Bethe-Salpeter equation for the bound states of two massive spinless bosons produced by the exchange of a massless spinless boson:

$$\phi(p)=G_a^{(1)}(p_a)G_b^{(1)}(p_b)-\frac{\lambda}{\pi^2}\int d^4k\frac{i\phi(k)}{-(p-k)^2+i\epsilon}, \quad (1)$$

where

$$p_a=m_a\eta+p\,,\qquad p_b=m_b\eta-p\,, \quad (2)$$

and η is defined in terms of the total four-momentum of the bound state P by

$$P=(m_a+m_b)\eta\,.$$

From its definition,

$$-\eta^2<1. \tag{3}$$

In Eq. (1), $G_i^{(1)}$ is the lowest-order propagator of a scalar boson of mass m_i, and λ is the coupling constant. We will henceforth take $m_a=m_b=1$. This restriction is not significant; Cutkosky solved the equation for $m_a\neq m_b$ and found that the spectrum had the same degeneracy structure. With this restriction, the Bethe-Salpeter equation becomes

$$[(p+\eta)^2+1][(p-\eta)^2+1]\phi(p) = -i\frac{\lambda}{\pi^2}\int d^4k\frac{\phi(k)}{(p-k)^2-i\epsilon}. \tag{4}$$

Wick's[6] analytic continuation of the energy variable, $k_4\to ik_4$, $p_4\to ip_4$, $\eta_4\to i\eta_4$, transforms the equation into

$$[(p+\eta)^2+1][(p-\eta)^2+1]\phi(p)=\frac{\lambda}{\pi^2}\int d^4k\frac{\phi(k)}{(p-k)^2}. \tag{5}$$

The squares of four-vectors are taken with a Euclidean metric. This is the equation which Cutkosky solved.

We will pursue Cutkosky's solution up to the display of the symmetries of the equation. We begin by parametrizing the Euclidean momentum space in spherical polar coordinates $|p|, \omega, \theta, \phi$, and then stereographically map the full momentum space onto the surface of a sphere of radius $\frac{1}{2}p_0$ in five dimensions by defining the angle ζ by

$$p_0\tan\tfrac{1}{2}\zeta=|p|. \tag{6}$$

The four angles ζ, ω, θ, ϕ parametrize the surface of the four-sphere onto which the momentum space has been mapped. It is convenient to introduce rectangular coordinates ξ_i in the five-dimensional space. The coordinates of points on the sphere are then

$$\xi_5=\tfrac{1}{2}p_0\cos\zeta,$$
$$\xi_4=\tfrac{1}{2}p_0\sin\zeta\cos\omega,\ \text{etc.}$$

The rectangular coordinates of the point onto which the momentum p has been mapped are

$$\xi_5=\tfrac{1}{2}p_0\frac{1-(p^2/p_0^2)}{1+(p^2/p_0^2)},\qquad \xi_\mu=\frac{p_\mu}{1+(p^2/p_0^2)}. \tag{7}$$

We now choose

$$p_0=(1+\eta^2)^{1/2},$$

and rewrite the Wick-rotated Bethe-Salpeter equation in terms of the variables ξ_i. From Eqs. (6) and (7), and the definition of a five-dimensional angular-integration element by

$$d^5\xi'=|\xi'|^4d|\xi'|\,d\Omega_{(5)}', \tag{8}$$

Eq. (5) becomes

$$[p_0^4-4(\eta\cdot\xi)^2]\tilde{\phi}(\xi)=\frac{\lambda}{8\pi^2}\int\frac{p_0^2d\Omega_{(5)}'\tilde{\phi}(\xi')}{1-\hat{\xi}\cdot\hat{\xi}'}. \tag{9}$$

Here $\tilde{\phi}=[\sec^6(\frac{1}{2}\zeta)]\phi$, $\hat{\xi}$ and $\hat{\xi}'$ are unit five-vectors directed along ξ and ξ', the center dot indicates a five-dimensional Euclidean inner product, and η has been extended to a five-vector by the definition $\eta_5\equiv0$.

It is in this form that we will discuss the symmetries of Cutkosky's equation, but we should first note two restrictions. The entire momentum space has been mapped onto the surface of the sphere $\xi\cdot\xi=(\frac{1}{2}p_0)^2$, and η_5 has been taken to be zero. Thus any transformation which mixes vectors ξ lying on this sphere with others not lying on it, or any transformation which causes η_5 to become nonzero, is not interpretable as a transformation on the original equation (5). We may not consider such transformations as possible symmetries.

Equation (9) is invariant under Lorentz transformations, which take the form of simultaneous rotations of ξ and η, leaving their fifth components unaffected. Additionally, it is invariant under all rotations of ξ alone which do not alter the η component of ξ, i.e., which leave $\eta\cdot\xi$ fixed. We will call these latter rotations Cutkosky transformations. This does not completely describe the symmetry group of the equation, however, since it is clear that a product of two transformations, one a Lorentz transformation and the other a Cutkosky transformation, is neither a pure Lorentz transformation nor a pure ξ-space rotation. We may put the problem we wish to solve as follows: Describe the structure of the full symmetry group of Cutkosky's equation.

By the full symmetry group we do not mean the group of all transformations which leave Eqs. (8) or (9) invariant, for this will surely be an uninteresting infinite-parameter group. The group we will seek is rather the smallest finite-parameter subgroup of the large group which still contains all the Lorentz and Cutkosky transformations. This group will ensure that the bound states have the $O(4)$ degeneracy found by Cutkosky, but will contain the minimum of extraneous content beyond this. It is thus the most suitable object for investigating how this degeneracy squares with the "no-go" theorems.

We may express the group-theoretic problem somewhat differently. We know the symmetry group of Cutkosky's equation in any given rest frame, but not the effect of transformations defined in one frame on solutions of the equation in a different frame. There are clearly an infinite number of ways of extending the definitions of symmetry transformations within the constraints set by the symmetry of the equation. For

[6] G. C. Wick, Phys. Rev. 96, 1124 (1954).

example, we could say that a transformation defined in one frame leaves solutions in all other frames unchanged. This, of course, is an infinite-parameter symmetry group (the parameters in each rest frame are independent). Our problem is to extend the definition of the symmetry transformations so that the full $O(4)$ invariance of the Cutkosky equation in every frame is included in the symmetry group, but we will want to find the extension which yields the minimal symmetry group.

III. GENERATORS OF THE SYMMETRY GROUP

In order to investigate the structure of the group of symmetry transformations on Cutkosky's equation, we will write the generators of such transformations explicitly, and by direct commutation discover the structure of the group. From the outset we will limit our consideration to generators which can be written in the form of Eq. (10), i.e., those containing one derivative only. The reason for this is that generators which are linear neither in the coordinate nor the derivative will tend, upon commutation, to yield generators with progressively larger numbers of derivatives and powers. Furthermore, requiring the generators to be linear in the coordinate will yield exactly the same structures as will requiring linearity in the derivative, and so the latter possibility is general enough.

The analysis then proceeds by exploiting Lorentz invariance. We classify generators by their transformation properties under the Euclidean $O(4)$ group corresponding to Lorentz transformations, and use the explicit form of generators transforming according to given irreducible representations to calculate the commutator of two such generators. This allows us to read off the structure constants of the group. We find that a finite-parameter group exists, and it is likely that it is the only one. It is certainly the smallest possible structure.

We consider generators of coordinate transformations in the ξ and η spaces which are of the form

$$G=\xi A\,\partial/\partial\xi+\eta B\,\partial/\partial\eta\,, \tag{10}$$

where A is a 5×5 matrix and B is a 4×4 matrix. We will consider transformations which leave Cutkosky's equation invariant, generate transformations on the physical momentum space, and turn solutions with mass m into others with the same mass. The generators of such transformations must, then, commute with $\xi\cdot\xi$ and η^2. The presence of the $\hat{\xi}\cdot\hat{\xi}'$ in Eq. (9) implies that A must be an antisymmetric matrix with no functional dependence on ξ.

The generators of the Euclidean $O(4)$ transformations which correspond to Lorentz transformations, G_ω, are characterized by constant matrices A and B, with $A_{\mu5}=0$ and $A_{\mu\nu}=B_{\mu\nu}=\omega_{\mu\nu}$; ω is an antisymmetric imaginary matrix, so that G_ω is Hermitian. We may classify all generators of the symmetry group according to their transformation properties under the Euclidean $O(4)$ group. It is sufficient to consider generators which belong to irreducible representations at $O(4)$, since the linear independence of inequivalent representations of $O(4)$ and the linearity of the generator algebra assure that if a generator which is the sum of terms transforming according to different irreducible representations occurs in the algebra, then each term must occur separately. The statement that G^{jm} transforms according to the j representation of $O(4)$ takes the form

$$(1/i)[G_\omega,G^{jm}]=\delta_\omega G^{jm}=T_{m'm}{}^{(j)}(\omega)G^{jm'}. \tag{11}$$

The implications of this equation for the matrices A^{jm} and B^{jm} characterizing G^{jm} are that $A_{\mu5}{}^{jm}$ is a vector four-dimensional spherical harmonic transforming according to the j representation of $O(4)$, while $A_{\mu\nu}{}^{jm}$ and $B_{\mu\nu}{}^{jm}$ are tensor spherical harmonics transforming according to the same representation.

In order to generate invariances of Eq. (9), G must commute with the $\xi\cdot\eta$ term in that equation. For the matrices A and B, this implies that

$$\eta_\mu A_{\mu5}\xi_5+\eta_\mu(A_{\mu\nu}-B_{\mu\nu})\xi_\nu=0. \tag{12}$$

Since A has no ξ dependence, if we are to retain a finite number of generators, $\eta_\mu(A_{\mu\nu}-B_{\mu\nu})$ must be proportional to ξ_ν. This, the antisymmetry of A, and the fact that G commutes with η^2 implies that

$$\eta_\mu A_{\mu\nu}=\eta_\mu B_{\mu\nu}. \tag{13}$$

Hence the matrix B may be replaced by the first four rows and columns of A. The generator G is then entirely characterized by the matrix A. This has the effect of transforming Eq. (12) into a transversality condition on $A_{\mu5}$:

$$\eta_\mu A_{\mu5}=0. \tag{14}$$

Because the only functional dependence in A is on η, we can construct the tensor spherical harmonics $A_{\mu5}$ and $A_{\mu\nu}$ by summing products of $O(4)$ spherical harmonics of $\hat{\eta}$ with fixed numerical tensors transforming according to the $(\frac{1}{2},\frac{1}{2})$ and $(1,0)\oplus(0,1)$ representations, respectively, weighted by the appropriate Clebsch-Gordan coefficient. $O(4)$ spherical harmonics can belong only to representations of the type (L,L), and so tensor harmonics transforming according to the $j=(j_+,\,j_-)$ representation can be constructed only for $|j_+-j_-|\leq1$. When the generators G are constructed in this way, their commutators can be evaluated from a knowledge of the effects of generators of $O(4)$ on the spherical harmonics and the decomposition of products of spherical harmonics.

The result is that in general, a set of generators, or even a single one with its $O(4)$ partners, will produce, upon commutation, an unending set of further generators transforming according to higher and higher representations of $O(4)$. For example, a generator with $A_{\mu5}{}^{(1/2,1/2)}$ nonvanishing and satisfying the transversality condition produces generators with nonvanishing $A_{\mu\nu}^{(1,0)}$, $A_{\mu\nu}^{(0,1)}$, $A_{\mu 5}^{(3/2,1/2)}$, $A_{\mu 5}^{(1/2,3/2)}$, $A_{\mu 5}^{(3/2,3/2)}$, etc. This set will not include all possible generators, but will include an infinite subset of them. Other infinite subsets can also be found. Thus, in general, we will have to deal with infinite-parameter groups. But since, as was remarked earlier, we are primarily interested in finite-parameter groups, we would like the set of generators produced by repeated commutation to close with a finite number of elements.

We can achieve this by taking advantage of the fact that the representation (j_+, j_-) according to which a generator transforms must satisfy $|j_+-j_-|\leq 1$. Thus if we demand $j_-=0$, then the commutator will only produce elements with $j_-=0$, and the fact that j_+ is bounded by 1 ensures that the generators close to form a finite set. Furthermore, generators transforming according to the (0,0) representation do not exist, because an $A_{\mu\nu}^{(0,0)}$ cannot be constructed, while an $A_{\mu 5}^{(0,0)}$ must violate the transversality condition (14). Thus the $O(4)$ Euclidean transformations, along with generators transforming to the (1,0) representation of $O(4)$, generate a finite-parameter group which will include the Cutkosky transformations. There is complete symmetry between j_+ and j_-, and so the (1,0) transformations can be replaced by (0,1) transformations to yield an equivalent group. We will now investigate in detail the structure of these finite-parameter groups.

IV. FINITE-PARAMETER SYMMETRY GROUP

The easiest way to find the structure of the group generated by the (1,0) operators is to construct the generators and evaluate their commutators. It will also be easy to see how the symmetry group changes when the Bethe-Salpeter equation is analytically continued back to Minkowski space.

We will denote the η-dependent transformations connecting ξ_μ with ξ_ν by C_i, and those connecting ξ_μ with ξ_5 by D_i. The latter may be constructed from the matrices χ_i which appear in the (1,0) generators of the $O(4)$ Euclidean group

$$J_i^+=\tfrac{1}{2}(\xi\chi^i\partial/\partial\xi+\eta\chi^i\partial/\partial\eta). \tag{15}$$

These matrices have the same algebra as the σ matrices,

$$\chi^i\chi^j=\delta_{ij}+i\epsilon_{ijk}\chi^k, \tag{16}$$

and commute with the matrices $\bar{\chi}^i$ which appear in the (0,1) generators of the Euclidean group J_i^-. The (1,0) generators D_i are

$$D_i=\xi_5\hat{\eta}\chi^i\partial/\partial\xi-\hat{\eta}\chi^i\xi\partial/\partial\xi_5, \tag{17}$$

where ξ, $\partial/\partial\xi$, and $\hat{\eta}$ are to be understood as four-dimensional row or column vectors according to their position with respect to χ^i. We know that the commutator of two D's must be a C, and because of the antisymmetry of the commutator we may write

$$(1/i)[D_i,D_j]=\epsilon_{ijk}C_k, \tag{18}$$

which defines the generators C_i. Note that C_i is quadratic in $\hat{\eta}$, and that because of the antisymmetry of χ^i, the tensor spherical harmonic associated with C_i is transverse to $\hat{\eta}$. Using the explicit forms for C_i and D_i and the algebra of the χ's [Eq. (16)], we find the remaining commutators:

$$\begin{aligned}(1/i)[C_i,C_j]&=\epsilon_{ijk}C_k,\\(1/i)[C_i,D_j]&=\epsilon_{ijk}D_k.\end{aligned} \tag{19}$$

These commutation relations show that the C_i and D_i generators form an $O(4)$ algebra whose commuting $O(3)$ factors are

$$K_i^\pm=\tfrac{1}{2}(C_i\pm D_i). \tag{20}$$

We may verify by using Eq. (15) that each set of generators C_i, D_i, and $K_i^\pm$ transform among themselves under the Euclidean group according to the (1,0) representation, i.e.,

$$\begin{aligned}(1/i)[J_i^+,C_j]&=\epsilon_{ijk}C_k,\\(1/i)[J_i^-,C_j]&=0,\ \text{etc.}\end{aligned} \tag{21}$$

Thus the structure of the full symmetry group, excepting the translations, is that of a semidirect product of $O(4)$ with itself, $O(4)\times O(4)$. The first factor is the Euclidean group. The structure of the group can also be regarded as a direct product, $O(4)\otimes(4)$, but the $O(4)$ group corresponding to Lorentz transformations is not a direct factor. The four commuting $O(3)$ factors are generated by $\{J_i^+-C_i\}$, $\{J_i^-\}$, $\{K_i^+\}$, and $\{K_i^-\}$.

In order to examine the Euclidean space equivalent of the little group for fixed momentum, we use the usual generators of the Euclidean $O(4)$ group, L_i and N_i, rather than the commuting combinations of them,

$$J_i^\pm=\tfrac{1}{2}(L_i\pm N_i). \tag{22}$$

The little group for states at rest is generated by $\{L_i\}$, $\{C_i\}$, and $\{D_i\}$, each evaluated at $\hat{\eta}=(0,0,0,1)$. C_i and D_i transform under L_i the same as they do under J_i^+, and since the L_i form an $O(3)$ algebra, the little group appears to be another semidirect product, $O(3)\times O(4)$. As before, a direct-product structure can be exhibited. $\{L_i-C_i\}$ and $\{C_i,D_i\}$ generate commuting $O(3)$ and $O(4)$ factors of the little group for $\hat{\eta}_4=1$. Since the χ^i's and $\bar{\chi}^i$'s are known matrices, we can explicitly evaluate the generators. They are functions of $\hat{\eta}$, and at $\hat{\eta}_4=1$, L_i-C_i vanishes. So although the little group is $O(3)\otimes O(4)$, only the $O(4)$ factor, the group generated by $\{C_i\}$ and $\{D_i\}$, is represented nontrivially. On states at rest, all elements of the $O(3)$ factor are represented by unity. Of course, L_i-C_i is not identically zero, but neither is it part of the little group for other momenta. For other momenta, the L_i are replaced by linear combinations of themselves and the N_i, which are cancelled by C_i evaluated at the relevant momentum.

That the faithfully represented little group is $O(4)$ is precisely Cutkosky's original observation, which is the same as the nonrelativistic result. The significance of the extra $O(3)$ group which drops out of the little group as if by magic is also clear. It is the relativistic analog of the group of rotations of the center of mass in the nonrelativistic problem. There, in the rest frame, the wave function of the center of mass is a zero three-momentum eigenstate, and so is rotationally invariant, corresponding to the vanishing of the rotation generators. Of course, when the entire system is moving, the center-of-mass rotations are not trivial, which corresponds to the fact that $L_i - C_i$ is not identically zero, but only vanishes for one $\hat{\eta}$. Nonrelativistically, where the separation of relative and total angular momentum is invariant, the groups of transformations on the relative and center-of-mass coordinates are independent. Here they are mixed up with one another, for the generator $L_i - C_i$ generates transformations on both ξ and η. But the differences are just the ones to be expected when one goes to a relativistically invariant description; all the general features of the nonrelativistic symmetry emerge, in fact, at least in the analytically continued Bethe-Salpeter equation.

V. SYMMETRY GROUP IN MINKOWSKI SPACE

All of our analysis of the symmetry group of Cutkosky's equation has been concerned with the analytically continued form, Eq. (8), in which the Lorentz metric $(+++-)$ is replaced by a Euclidean one. We represented the generators of the symmetry group as differential operators acting on a Hilbert space of functions of two Euclidean four-vectors ξ_μ and η_μ and a scalar ξ_5. We must now examine how the symmetry group is modified when the equation is analytically continued back to its initial form, since it is the Minkowski-space equation with which we are actually concerned. We shall write down the generators of analytically continued transformations on the Minkowski-space variables and investigate the structure of the group they generate. Then we will show directly that these symmetry transformations do, in fact, leave the Minkowski-space Bethe-Salpeter equation invariant.

The original analytic continuation of the Bethe-Salpeter equation was effected by the replacement of the fourth component of Minkowski-space four-vectors p_4 by ip_4. If we define the matrix

$$\mathcal{A} = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & i \end{bmatrix}, \tag{23}$$

then the reverse continuation on η is

$$\eta \to \mathcal{A}^{-1}\eta, \tag{24}$$

and the Lorentz metric $(+++-)$ is restored. The replacement on ξ implied by the analytic continuation of p is

$$\xi \to \mathcal{A}^{-1}\xi, \tag{25}$$

where $\mathcal{A}$ has been extended to a 5×5 diagonal matrix with $\mathcal{A}_{55}=1$. Five-vector inner products are taken with the metric $(+++-+)$. We will denote both the 4×4 and 5×5 matrices by $\mathcal{A}$ and the four- and five-dimensional metrics by g; the proper dimensionality will be indicated by the dimensionality of the vectors multiplied by the matrix.

The generators of the transformations on the analytically continued variables are obtained from the generators of Sec. IV when expressed in the form Eq. (10), by the replacement

$$G \to \xi g\mathcal{A}A\mathcal{A}\partial/\partial\xi + \eta g\mathcal{A}B\mathcal{A}\partial/\partial\eta. \tag{26}$$

(All appearances of the metric in the generators will be explicit.) If we define the matrices

$$\begin{aligned} X^i &= \mathcal{A}\chi^i\mathcal{A}, \\ \bar{X}^i &= \mathcal{A}\bar{\chi}^i\mathcal{A}, \end{aligned} \tag{27}$$

then the generators of Lorentz transformations may be written

$$\begin{aligned} J_i^+ &= \tfrac{1}{2}(\xi gX^i\partial/\partial\xi + \eta gX^i\partial/\partial\eta), \\ J_i^- &= \tfrac{1}{2}(\xi g\bar{X}^i\partial/\partial\xi + \eta g\bar{X}^i\partial/\partial\eta), \end{aligned} \tag{28}$$

where in the above and henceforth, ξ without an index denotes the first four components only. The real generators of Lorentz transformations are defined by

$$J_i^\pm = \tfrac{1}{2}(L_i \pm iN_i). \tag{29}$$

The operators L_i and N_i, and not $J_i^\pm$, are supposed to be Hermitian. That they are can be seen by their explicit construction from the X^i and $\bar{X}^i$ matrices.[7]

The algebra of the X^i matrices, which follows from that of the χ^i's, is

$$\begin{aligned} X^i gX^j &= g\delta_{ij} + i\epsilon_{ijk}X^k, \\ \bar{X}^i g\bar{X}^j &= g\delta_{ij} + i\epsilon_{ijk}\bar{X}, \\ X^i g\bar{X}^j - \bar{X}^j gX^i &= 0. \end{aligned} \tag{30}$$

These relations allow us to calculate the commutators of Minkowski-space generators. We can verify at once that $J_i^\pm$ are two commuting triplets of operators, each of which generates an $O(3)$ group, and that L_i and N_i have their proper $O(3,1)$ commutation relations.

Using the general prescription (26), we can write the D_i's, the generators of the (1,0) transformations connecting ξ_5 to ξ_μ, as operators acting on functions of the Minkowski-space variables:

$$D_i = \xi_5\hat{\eta}gX^i\partial/\partial\xi + \xi gX^i\hat{\eta}\partial/\partial\xi_5. \tag{31}$$

[7] Hermiticity is defined here with respect to the metric $(\psi,\phi) \equiv \int d^4p\, \psi^*(p)\phi(p)$. We show in the Appendix that this is equivalent to Hermiticity in the field-theory Hilbert space.

Since η is timelike for physical values of the momenta, $\hat{\eta}$ is normalized to $\hat{\eta}^2=\hat{\eta}g\hat{\eta}=-1$. Similarly the C_i's, the generators of (1,0) transformations not involving η or ξ_5, are obtained from their Euclidean-space forms. In order that they retain their usual $O(3)$ commutation relations, we multiply them by -1. The operators L_i, N_i, C_i, D_i are generators of the Minkowski-space symmetry group. Their algebra, which we now calculate, determines the symmetry-group structure.

The commutators of the C_i and D_i generators follow from Eq. (30), and are

$$(1/i)[C_i,C_j]=\epsilon_{ijk}C_k\,,\quad (1/i)[C_i,D_j]=\epsilon_{ijk}D_k\,,\quad (1/i)[D_i,D_j]=-\epsilon_{ijk}C_k\,, \tag{32}$$

which are the commutation relations of $O(3,1)$. Each of the triplets has the same commutation relations with $J_i^{\pm}$:

$$(1/i)[J_i^+,C_j]=\epsilon_{ijk}C_k\,,\quad (1/i)[J_i^-,C_j]=0\,. \tag{33}$$

These show that C_i and D_i are each (1,0) triplets under the Lorentz group. The (1,0) representation of the Lorentz group is complex, though of course that of $O(4)$ is not, and the above commutation relations show that C_i and D_i cannot be Hermitian operators. Their commutation relations with the Hermitian generators of the Lorentz group are

$$(1/i)[L_j,C_j]=\epsilon_{ijk}C_k\,,\quad (1/i)[N_j,C_j]=-i\epsilon_{ijk}C_k\,, \tag{34}$$

with D_i satisfying the same relations. The extra factor of i in the second commutator, besides verifying that C_i and D_i are not Hermitian, means that in addition to C_i and D_i, the generators iC_i and iD_i occur in the algebra of the Minkowski-space symmetry group. The four triplets of operators C_i, D_i, iC_i, iD_i generate the four-dimensional complex rotation group $O(4,C)$, and since the Lorentz-group generators turn these into themselves, the structure of the full symmetry group, excepting the translations, is a semidirect product of the Lorentz group with $O(4,C)$, i.e., $O(3,1)\times O(4,C)$. As in the Euclidean case, we could replace J_i^+ by $J_i^+-C_i$ to display the structure formally as $O(3,1)\otimes O(4,C)$, but in this case, the $O(3,1)$ factor is generated by non-Hermitian operators. In the semidirect-product form, the $O(3,1)$ group has Hermitian generators. The $O(4,C)$ group, of course, does not.

The little group for fixed momentum of this symmetry group clearly includes the full $O(4,C)$ group, and we know that the little group of the Lorentz group is $O(3)$, so that the little group has the structure $O(3)\times O(4,C)$, since any subgroup of the Lorentz group is also a group of linear transformations on the elements of the $O(4,C)$ algebra. However, as we have noted several times, the elements of the $O(4,C)$ algebra are non-Hermitian operators, and so generate nonunitary transformations on the Bethe-Salpeter wave functions. These transformations are symmetries of the equation—they turn solutions of the (homogeneous) equation into other solutions—but they do not preserve the normalization of the solutions. Since these transformations will change probabilities, we cannot interpret them as symmetries of a quantum-mechanical system in the usual way. If, however, we fix our attention to bound states of a given definite four-momentum, we can find a subalgebra of the $O(4,C)$ algebra which is composed of operators which, when restricted to that subspace of states, are Hermitian.

To see this, and to discover the structure of the corresponding group of unitary transformations, we fix our attention on states at rest, $\hat{\eta}=(0,0,0,1)$. Using Eqs. (31) and (32), we can explicitly construct C_i and D_i for this momentum, and when this is done we see that C_i and iD_i are Hermitian operators. The commutation relations, Eq. (32), show that these operators form an $O(4)$ algebra. The existence of this algebra of Hermitian operators on states at rest depends on the enlargement of the $O(3,1)$ algebra to $O(4,C)$, which was required by the commutation relations involving the Hermitian generators of the Lorentz group, Eq. (34). Since the little group of the Lorentz group for states at rest is the rotation group $O(3)$, also generated by Hermitian operators, the unitary little group for states at rest has the structure $O(3)\times O(4)$. The structures of the unitary little groups are the same in all frames because they are related by unitary similarity transformations.

A separation of the unitary little group for states at rest into direct factors is quite meaningful, since all the generators are Hermitian (on these states). The commutation relations, Eqs. (33) and (34), show that the triplet L_i-C_i generates an $O(3)$ group which commutes with the $O(4)$ generated by C_i and iD_i, so that the group has the structure $O(3)\otimes O(4)$, where both factors, restricted to states at rest, are unitary. As in the Euclidean case, the $O(3)$ factor vanishes when restricted to states at rest. And so we are led, finally, to the structure of the unitary little group of the full symmetry group for states at rest; it is $O(3)\otimes O(4)$, where the $O(3)$ factor is represented by unity. The faithfully represented little group is precisely $O(4)$. This group fulfills all the requirements of a quantum-mechanical symmetry group of the bound states for fixed momentum. Besides not changing the four-momentum, it is a symmetry group of the Bethe-Salpeter equation, and is unitary on the given states. This group is the relativistic analog of the $O(4)$ symmetry group of the nonrelativistic hydrogen atom, and the physical explanation of the $O(3)$ factor which is represented trivially as the group of rotations of the center of mass, which was given in Sec. IV, applies more properly here.

C_i and D_i have derivatives in ξ only, and so commute with the translations, which are represented as just the multiplicative operators $2\ \eta_\mu$. The full symmetry group, including the Poincaré group, is then

$$\mathcal{G}=O(3,1)\times[O(4,C)\otimes T_4]. \tag{35}$$

The Abelian group T_4 contains the translations.

The foregoing discussion of the structure of the Minkowski-space symmetry group—the enlargement of the semidirect factor from $O(4)$ to $O(4,C)$, the identification of the nonunitary and unitary little groups, and the nonunitarity of the elements of the full group—all follow from the explicit forms of the generators C_i and D_i. These generators are, however, the analytic continuations of the generators of the symmetry group of the analytically continued equation, and so we will show directly that they generate symmetries of the original Bethe-Salpeter equation, Eq. (4). This will show that the structure of the symmetry group is independent of the analytic continuation, as well as validating the double-analytic-continuation technique we have used to discover it.

The technique will be to map the Minkowski-space Bethe-Salpeter equation onto a surface in a five-dimensional space in such a way as to make the higher symmetry of the equation manifest. To this end, keeping $m_a=m_b=1$, we define

$$\begin{aligned}\eta&=(\text{total four-momentum})/(m_a+m_b)\,,\\ p_0{}^2&=1+\eta^2\,,\\ \Lambda&=1+p^2/p_0{}^2\,.\end{aligned} \tag{36}$$

In contrast to the Euclidean case where η^2 was positive and $p_0{}^2$ greater than 1, here η^2 is negative, but since it is always greater than -1, p_0 remains real. The four-vector p can take both spacelike and timelike values, so that the sign of Λ is not fixed. By analogy to the construction of ξ from p [Eq. (7)], we define

$$\begin{aligned}\xi_\mu&=p_\mu/\Lambda\,,\\ \xi_5&=\tfrac{1}{2}p_0(-\Lambda+2)/\Lambda\,.\end{aligned} \tag{37}$$

This defines the mapping of the Bethe-Salpeter equation into the ξ space. The physical subspace satisfies $\xi\cdot\xi=(\frac{1}{2}p_0)^2$. [Five-dimensional inner products are taken with the metric $(+++-+)$.] This is a spacelike, or one-sheeted, hyperboloid in the five-dimensional space. We define the invariant "angular" integration element $d\Omega_{(5)}$ by

$$d^5\xi=|\xi|^4d|\xi|\,d\Omega_{(5)}\,, \tag{38}$$

and $\tilde{\phi}=\Lambda^3\phi$. The Bethe-Salpeter equation becomes

$$[p_0{}^4-4(\xi\cdot\eta)^2]\tilde{\phi}(\xi)=-i\frac{\lambda}{8\pi^2}\int\frac{p_0{}^2d\Omega_{(\sigma)}{}'\tilde{\phi}(\xi')}{1-\hat{\xi}\cdot\hat{\xi}'}\,. \tag{39}$$

In this form the Bethe-Salpeter equation is manifestly invariant under $O(4,1)$ transformations in the ξ space which leave the component of ξ lying along η unchanged. The generators of such transformations may be written

$$G=\xi gT\partial/\partial\xi\,, \tag{40}$$

where the 5×5 matrix T is antisymmetric and satisfies the transversality condition $\eta gT=0$. The quantities C_i and D_i, as defined earlier, satisfy these restrictions, and so generate symmetries of the equation. The Bethe-Salpeter equation is also invariant under Lorentz transformations, and so we could carry through the complete discussion of the symmetry groups without ever making an analytic continuation to Euclidean space variables. Note that no reality condition is imposed on T; the invariance of the equation follows from the antisymmetry alone. The Hermiticity of the resulting operators, and the possibility of their interpretation as generators of symmetries of a quantum-mechanical system, is what is affected by the reality properties of T. The same is true in the Euclidean-space case, where the $O(4)$ symmetry group could have been enlarged to $O(4,C)$ by allowing iC_i and iD_i to be part of the generator algebra. The only effect would be to include extra, nonunitary elements in the symmetry group. The unitary part of the symmetry group, the only part allowing a quantum-mechanical interpretation, would have been unchanged.

VI. CONTINUUM STATES

In this section we will examine the symmetry group of the continuum states, and show that the Bethe-Salpeter equation has the same spectral degeneracy as nonrelativistic Coulomb scattering. Following the discussion of the Minkowski-space bound states, we will construct from the Minkowski-space generators new ones which are appropriate to the unbound states, and then show that these do generate symmetries of the scattering-state Bethe-Salpeter equation.

Since p_0, as defined in Eq. (36), is imaginary for the unbound states, the mapping (37) is not real. We can correct this by the replacement

$$\xi\to\mathcal{B}\xi \tag{41}$$

with

$$\mathcal{B}=\begin{bmatrix}1&&&&0\\&1&&&\\&&1&&\\&&&1&\\0&&&&i\end{bmatrix}.$$

The new mapping $p\to\xi$ is real for unbound states. Defining

$$\gamma=\mathcal{B}^2\,,$$

we may transcribe the generators to the new variables

$$G\to\xi\gamma g\mathcal{B}\mathcal{A}A\,\mathcal{A}\mathcal{B}\partial/\partial\xi+\eta g\mathcal{A}B\mathcal{A}\partial/\partial\eta\,. \tag{42}$$

$\mathcal{B}$ and γ affect only the fifth component ξ, so that there is no need to reexamine the construction of the generators from four-vectors. Also, those generators which do not involve ξ_5 are unchanged, so that only D_i is changed. According to Eqs. (31) and (42), it becomes

$$D_i = -i(\xi_5 \hat{\eta} g X^i \partial/\partial\xi - \xi g X^i \hat{\eta} \partial/\partial\xi_5). \quad (43)$$

Therefore all the commutation relations, Eqs. (32)–(34), still hold, and the full symmetry group, excepting the translations, generated by $\{L_i, N_i, C_i, D_i, iC_i, iD_i\}$, is $O(3,1)\times O(4,C)$, and the little group for fixed momenta is $O(3)\times O(4,C)$. The possibility of replacing the semidirect product by a direct product, at the cost of the unitarity of the $O(3,1)$ factor, exists here also. Each of the elements of the $O(4,C)$ group are nonunitary operators. Since the translations commute with the $O(4,C)$ factor, the structure of the full symmetry group is $O(3,1)\times[O(4,C)\otimes T_4]$.

The difference between the bound and continuum states appears when we examine the unitary part of the little group. The Hermiticity of L_i and C_i, when restricted to states at rest, is of course not altered. But D_i acquired an extra factor of $-i$, and thus is Hermitian when restricted to states at rest. Therefore, the commutation relations, Eqs. (32) and (33), show that the little group of states at rest, generated by $\{L_i, C_i, D_i\}$ is $O(3)\times O(3,1)$. Neither L_i nor C_i is changed; therefore, when the little group is factored into the direct product $O(3)\otimes O(3,1)$, the $O(3)$ factor is still represented by unity. For the unbound states, the nontrivially represented unitary little group for states at rest is $O(3,1)$, exactly the nonrelativistic result. The significance of the extra $O(3)$ factor is as before.

To demonstrate that the above operators do generate symmetries of the Bethe-Salpeter equation, we will follow the same procedure as for the bound states. We map the equation onto a hyperboloid in a five-dimensional ξ space, the mapping being given by Eqs. (37) and (41). If dot products of five-vectors are taken with metric $(+++--)$, the continuum state Bethe-Salpeter equation becomes

$$[p_0^4 - 4(\xi\cdot\eta)^2]\tilde{\phi}(\xi) = -i\frac{\lambda}{8\pi^2}\int\frac{p_0^2 d\Omega_{(5)}'}{1-\xi\cdot\xi'}\tilde{\phi}(\xi). \quad (44)$$

This equation is manifestly invariant under all transformations on ξ which leave their dot products invariant, and which leave $\xi\cdot\eta$ unchanged. Such transformations are generated by operators of the form

$$G = \xi(g\gamma)T\partial/\partial\xi, \quad (45)$$

with T an antisymmetric 5×5 matrix satisfying

$$\eta(g\gamma)T = 0. \quad (46)$$

According to Eq. (42), C_i and D_i are of this form, with

$$T = \mathfrak{B}\mathfrak{A}A\mathfrak{A}\mathfrak{B}.$$

T is antisymmetric because A is, and satisfies the transversality condition because the X^i matrices are antisymmetric. Thus our description of the symmetry group of the continuum-state Bethe-Salpeter equation is correct.

VII. SYMMETRIES OF BOUND-STATE SCATTERING

Having investigated the symmetry group of the bound states in the Bethe-Salpeter equation, we will now examine the consequences of trying to extend the symmetry group to the scattering of two bound states. We will find an example of the fact that if bound states scatter, the S matrix cannot be invariant under a symmetry group which nontrivially combines Poincaré invariance with an internal symmetry.

We will extend the group to two-particle scattering in the usual way, by assuming that two-particle states transform under the symmetry group according to the direct product of the representations according to which the component single-particle states transform, and that the S matrix is invariant under the transformations of the symmetry group. With ordinary symmetry groups, whose elements are all unitary operators, this procedure yields selection rules and relations between observable amplitudes. However, because of the unusual unitarity properties of the group elements, we will discover that the above assumptions allow only forward scattering (and so, by analyticity, no scattering). The proof of this assertion does not depend on the detailed dynamics of the relativistic hydrogen atom, but only on certain general features of the symmetry group. We will state and prove the assertion without specific reference to the hydrogen atom.

We assume that the symmetry group of whatever system we are considering has the structure

$$\mathcal{G} = P\times K, \quad (47)$$

where the elements of K commute with the translations and are nonunitary operators, i.e., they transform physical states into other physical states with the same four-momentum, but do not preserve the norms of states. We further assume that for any given Lorentz frame there is a subgroup of K which, when restricted to one-particle states at rest in that frame, is represented unitarily, and, in addition, that the specification of the subgroup determines the frame. The intersection of the subgroups associated with two directions in space-time is the group of operators which are represented unitarily on states of a single particle at rest in either frame. We finally assume that the specification of this group uniquely determines the plane in four-momentum space spanned by the two particle momenta.

It is easy to see that the symmetry group of Cutkosky's Bethe-Salpeter equation has all these assumed properties. That it has the semidirect-product structure may be seen from Eq. (35). In this case, K is $O(4,C)$, all of whose generators we found were non-Hermitian. Furthermore, on states describing a single bound state at rest, the generators C_i, iD_i have Hermitian representations. On states moving with relativistic rapidity $\mathbf{u}$,[8] the generators with Hermitian representations are

$$\begin{aligned} C_i' &= e^{i\mathbf{u}\cdot\mathbf{N}} C_i e^{-i\mathbf{u}\cdot\mathbf{N}}, \\ iD_i' &= e^{i\mathbf{u}\cdot\mathbf{N}} iD_i e^{-i\mathbf{u}\cdot\mathbf{N}}. \end{aligned} \tag{48}$$

The Lie algebras spanned by C_i' and iD_i' for any two rapidities are distinct, and so the groups which are represented unitarily on one-particle states at rest in two different Lorentz frames are distinct. To find the intersection of two such groups, we transform to the center-of-mass system with both particles moving parallel to the three axis. The two sets $\{C_i', iD_i'\}$ are as above, but with rapidities of opposite sign. The only elements in both algebras are those in the algebra spanned by C_3 and iD_3, and so the intersection of the symmetry groups will be the Abelian group generated by this algebra. This group specifies in an obvious way the 34 plane, which is the plane spanned by the momenta of the two particles in their center-of-mass frame.

Given our assumptions, it is easy to see that only forward (or backward) scattering of bound states is allowed. We consider an incoming state of two particles, seen in their center-of-mass frame, called a and b:

$$|ab\mathbf{p}\text{ in}\rangle,$$

where $\mathbf{p}$ is the spatial momentum of a. Under K this state transforms according to the direct product of the representations of K on the states $|a\mathbf{p}\rangle$ and $|b-\mathbf{p}\rangle$. Since the direct product of two unimodular matrices is unitary only when the component matrices are, the only elements of K which are represented unitarily on $|ab\mathbf{p}\text{ in}\rangle$ are those which are represented unitarily on both $|a\mathbf{p}\rangle$ and $|b-\mathbf{p}\rangle$.

We have assumed K to be a symmetry group, so that its elements commute with the S matrix. So if k, an element of K, is unitary on a subspace of in states, it will also be unitary on the space obtained by applying the S operator to that subspace. Since by assumption k does not change either the number of particles or the momentum of any particle, it will be unitary on any space of two particle states with relative momentum $\mathbf{p}'$ for which

$$\langle a'b'\mathbf{p}'\text{ in}|S|ab\mathbf{p}\text{ in}\rangle \neq 0, \tag{49}$$

and on the component single-particle states $|a'\mathbf{p}'\rangle$ and $|b'-\mathbf{p}'\rangle$ also. By assumption, if we know the group that is represented unitarily on one-particle states of two different rapidities, then this knowledge specifies the plane spanned by their four-momenta. Hence the $\mathbf{p}E$ plane [the plane containing the four-momenta $(E, \mathbf{p})$ and $(E, -\mathbf{p})$] and the $\mathbf{p}'E$ plane [the plane containing $(E, \mathbf{p}')$ and $(E, -\mathbf{p}')$] must be the same, and so $\mathbf{p}$ and $\mathbf{p}'$ must be parallel. Hence we have proved that given our general assumptions about $\mathcal{G}$, only forward (or backward) elastic scattering is possible. Similarly, only inelastic scattering into states with any number of particles will be possible when all the outgoing particles have their spatial momenta aligned with those of the incoming pair of particles. However, since the S matrix is an analytic function of momentum, it cannot permit scattering in a single direction only, and so it must be the identity matrix.

The conclusion we are forced to from this result is simply that the sort of symmetry group we have been considering cannot be relevant to physical scattering processes. Although it may be perfectly sensible to classify single-particle states according to their transformation properties under this group, we cannot sensibly expect scattering amplitudes to be invariant under it, or under the unitary part of it.

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APPENDIX: UNITARITY OF THE CUTKOSKY OPERATORS

In Sec. V, we designated as Hermitian any symmetry generator of the form

$$C = (1/i)(\xi_i \partial/\partial\xi_j - \xi_j \partial/\partial\xi_i), \tag{A1}$$

and called the transformations generated by it unitary. However, we wish to retain the association between the solutions of the Bethe-Salpeter equation and the bound states in the Hilbert space of the field theory. By a unitary operator we ought to mean one which preserves the norms of states in the Hilbert space, and so we must show that the generator, Eq. (A1), is Hermitian with respect to the Hilbert-space norm. That is the aim of this appendix.

We will begin by deriving, from the normalization of the bound states in Hilbert space, a normalization condition on the Bethe-Salpeter wave functions.[9] An operator which is Hermitian in this norm generates unitary transformations on the Hilbert space, and we shall show C to be Hermitian in that norm.

The equation for the Green's function is

$$G = K^{-1} + K^{-1}IG \tag{A2a}$$

$$= K^{-1} + GIK^{-1}, \tag{A2b}$$

where K is the inverse of the free Green's function.

[8] The rapidity is defined as $\mathbf{u} \equiv (\mathbf{v}/|\mathbf{v}|) \tanh^{-1}|\mathbf{v}|$.

[9] R. E. Cutkosky and M. Leon, Phys. Rev. **135**, B1445 (1964).

Near a bound state of mass m, we have

$$G=i\sum_a \frac{\phi_a\bar{\phi}_a}{p^2-m^2}+\text{terms regular at}\quad p^2=m^2. \quad \text{(A3)}$$

To obtain this equation, we insert a complete set of states into the expression $T\langle 0|\psi\psi\psi\psi|0\rangle$ for the Green's function, so that Eq. (A3) has implicit in it the Hilbert-space normalization of the bound states. By inserting this into Eq. (A2) and looking near $p^2=m^2$, one gets the equations for the Bethe-Salpeter wave functions,

$$\phi_a=K^{-1}I\phi_a\,, \quad \text{(A4a)}$$

$$\bar{\phi}_a=\bar{\phi}_aIK^{-1}. \quad \text{(A4b)}$$

We extend these equations off their mass shells, i.e., allow values of P_μ for which $p^2\neq m^2$:

$$\psi_n=\lambda_nK^{-1}I\psi_n\,, \quad \text{(A5a)}$$

$$\bar{\psi}_n=\lambda_n\bar{\psi}_nIK^{-1}. \quad \text{(A5b)}$$

From these equations we obtain the orthogonality condition

$$\bar{\psi}_mK\psi_n=0 \quad \text{if} \quad \lambda_m\neq\lambda_n. \quad \text{(A6)}$$

We further orthogonalize the degenerate wave functions ($\lambda_m=\lambda_n$) by requiring that

$$\bar{\psi}_mK\psi_n=0 \quad \text{if} \quad m\neq n. \quad \text{(A7)}$$

We now wish to expand the Green's function in a series of solutions of Eq. (A5a), but because I is not Hermitian, these do not necessarily form a complete set. Hence we write

$$G=\sum_n \psi_n\pi_n+R. \quad \text{(A8)}$$

Since a subset of the solutions of Eq. (A5a), when $p^2=m^2$, are just the solutions of Eq. (A4a), we have

$$\lim_{p^2\to m^2}(p^2-m^2)R=0. \quad \text{(A9)}$$

Using the Green's function equation (A2a) and the orthogonality condition (A7), we may solve for π_n, obtaining

$$\pi_n=\frac{\bar{\psi}_n-\zeta_n}{(1-1/\lambda_n)\bar{\psi}_nK\psi_n}, \quad \text{(A10)}$$

where

$$\zeta_n=\bar{\psi}_nKR-\bar{\psi}_nIR=(1-1/\lambda_n)\bar{\psi}_nKR. \quad \text{(A11)}$$

The Green's function is, then,

$$G=\sum_n \frac{\psi_n(\bar{\psi}_n-\zeta_n)}{(1-1/\lambda_n)\bar{\psi}_nK\psi_n}+R. \quad \text{(A12)}$$

We now examine

$$\lim_{p^2\to m^2}(p^2-m^2)G=\epsilon\sum_{n(\lambda_n=1)}\frac{\bar{\psi}_n\psi_n}{\bar{\psi}_nK\psi_n}, \quad \text{(A13)}$$

where ϵ is the derivative

$$\epsilon=\lim_{p^2\to m^2}\frac{p^2-m^2}{1-1/\lambda}. \quad \text{(A14)}$$

Since, in the form of Eq. (A13), G is independent of the normalizations of the individual wave functions ψ_n, we choose them so that all the $\bar{\psi}_nK\psi_n$ factors are equal to the same constant c, so that

$$\lim_{p^2\to m^2}(p^2-m^2)G=\frac{\epsilon}{c}\sum_{n(\lambda_n=1)}\psi_n\bar{\psi}_n. \quad \text{(A15)}$$

The same quantity, calculated from Eq. (A3), is

$$\lim_{p^2\to m^2}(p^2-m^2)G=i\sum_a \phi_a\bar{\phi}_a. \quad \text{(A16)}$$

Note that the wave functions ψ_n and $\bar{\psi}_n$ in Eq. (A15), as well as the wave functions ϕ_a and $\bar{\phi}_a$, satisfy Eq. (A4), so that the wave functions are linear combinations of each other. Equation (A15) was derived from the assumption that

$$\bar{\psi}_nK\psi_m=c\delta_{mn}\,, \quad \text{(A17)}$$

and so the quadratic form, Eq. (A15), will be preserved by any symmetry transformation on the ψ_n wave functions which preserves that orthonormality condition. Equation (A16) was derived from the Hilbert-space normalization of the bound states, and so it will be invariant under unitary symmetry transformations on the Hilbert space. Thus we have two groups of transformations which preserve the same invariant quadratic form, and so these two groups of transformations must be the same. Thus any symmetry transformation on the wave functions ϕ_a which leaves p_μ unchanged and preserves

$$\bar{\phi}_aK\phi_b=c\delta_{ab} \quad \text{(A18)}$$

corresponds to a unitary transformation on the Hilbert space. We may use Eq. (A18) as the proper norm for states with the same p_μ, with respect to which we will determine the Hermiticity of C, Eq. (A1).

The norm

$$(\phi_a,\phi_b)=\int d^4p\,\bar{\phi}_a(p)K(p)\phi_b(p)$$

may be written as the ξ-space integral

$$(\phi_a,\phi_b)=\tfrac{1}{16}p_0^4\int d\Omega_{(5)}\bar{\phi}_a'\phi_b'[p_0^4-4(\xi\cdot\eta)^2], \quad \text{(A19)}$$

where $\phi'=\Lambda^3\phi$ [see Eq. (36)]. Symmetry transformations must leave $\xi\cdot\eta$ invariant, so we must have $[C,\xi\cdot\eta]=0$ for C to generate a symmetry.

In order to show that C is Hermitian in this norm, we need the relation between $\bar{\phi}$ and ϕ. Recall that in coordinate space,

$$\chi(x)=e^{-iP\cdot X}\langle 0|T\psi(x_1)\psi(x_2)|\alpha\rangle,$$
$$\bar{\chi}(x)=e^{iP\cdot X}\langle\alpha|T\psi(x_1)\psi(x_2)|0\rangle. \tag{A20}$$

From TP invariance (our fields are Hermitian), we see that[10]

$$\bar{\chi}(x)=e^{iP\cdot X}\langle\bar{\alpha}|\bar{T}\psi(-x_1)\psi(-x_2)|0\rangle^*,$$

where $\bar{\alpha}$ denotes the TP-reversed state to α. Assuming equal masses, and translating by $x_1+x_2=2X$, we get

$$\bar{\chi}(x)=e^{-iP\cdot X}\langle 0|T\psi(x_1)\psi(x_2)|\bar{\alpha}\rangle \tag{A21}$$

by the Hermiticity of the field ψ. Hence $\bar{\chi}$ is the wave function of the TP-reversed state to that for which χ is the wave function, and so $\bar{\phi}$ is obtained from ϕ by taking the momentum-space wave function of the TP-reversed state.

To find the effect of TP on C, note that C generates a real transformation on ξ space, and so, by Eq. (37) is equivalent to the generator of a real transformation on p space. Hence C can be written

$$C=(\text{real function of } p)_\mu \frac{1}{i}\frac{\partial}{\partial p_\mu}.$$

But under TP, $p\to p$, $(1/i)\partial/\partial p\to -(1/i)\partial/\partial p$, so that,

$$C\to -C. \tag{A22}$$

Therefore

$$\begin{aligned}(C\phi_a,\phi_b)&=\tfrac{1}{16}p_0^4\int d\Omega_{(5)}\bar{C}\bar{\phi}_a'\phi_b'[p_0^4-4(\Xi\cdot H)^2]\\&=-\tfrac{1}{16}p_0^4\int d\Omega_{(5)}(C\bar{\phi}_a')\phi_b'[p_0^4-4(\Xi\cdot H)^2]\\&=\tfrac{1}{16}p_0^4\int d\Omega_{(5)}\bar{\phi}_a'(C\phi_b')[p_0^4-4(\Xi\cdot H)^2]\\&=(\phi_a,C\phi_b).\end{aligned} \tag{A23}$$

Hence C is Hermitian in the (,) norm, and our use of its Hermiticity in the text is justified.

[10] $\bar{T}$ is the antitime ordering operator.